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RAYLEIGH–BENARD PROBLEM FOR AN ANOMALOUS FLUID

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The stability of the state of rest of a heated infinite horizontal layer of a viscous heat-conducting fluid (the Rayleigh–Benard problem) is considered. The equation of state for the fluid takes into account the nonmonotonic temperature and pressure dependence of water density. Instability of the mechanical equilibrium with respect to small monotonic perturbations is studied. The effect of the problem parameters on the Rayleigh numbers and their corresponding critical motions is investigated numerically using linear theory. Numerical investigation of the spectral problem is based on the Godunov–Abramov orthogonalization method. The calculation results are compared with the well-known results for the limiting case where the density is considered a quadratic function of temperature and does not depend on pressure.

Key words: *Rayleigh–Benard problem, Oberbeck–Boussinesq approximation, anomalous fluid, instability, perturbation monotonicity principle.*

Introduction. Interest in the Rayleigh–Benard problem for an anomalous fluid is motivated by the recent studies performed at Lake Baikal. The results of observations obtained using modern instrumentation indicate the existence of a deep mixing mechanism in the lake due to transport of Lake Baikal surface water to bottom regions [1, 2]. There are several hypotheses to explain this phenomenon. One of these hypotheses is based on the effect of anomalous thermal expansion of water.

If the pressure–density relation is ignored, the water density is a nonmonotonic function of temperature. This function reaches a maximum at a temperature approximately equal to 4°C (the so-called thermal-expansion inversion temperature). In this case, the equation of state for water can be written as

$$\rho = \rho_0[1 - \gamma(T - T_0)^2],$$

where ρ_0 is the maximum density, γ is the thermal-expansion coefficient, and T_0 is the inversion temperature. The nonmonotonic temperature dependence of the density is responsible for a complex stratification in a fluid layer whose surface temperature is higher than the inversion temperature and whose lower-boundary temperature is lower than the inversion temperature. Above the inversion point, the density gradient coincides with the gravity direction and the fluid is gravitationally stable. Below this point, the density decreases with increasing depth and the fluid stratification is unstable. Convective flows that arise in the lower unstable region propagate to the upper, stably stratified zone. Similar phenomena occur in other situations where stable fluid layers bound an unstably stratified region. Such convection is called a penetrative one. In some cases, however, one should take into account the effects caused by density deviations due to pressure variations. Vereshchagin was the first to pay attention to the importance of these factors [3]. For example, the water inversion temperature is not constant but decreases with increasing depth (and, hence, pressure) by approximately 0.21°C per each 100 m [2]. The maximum values of the density and thermal-expansion coefficient are also functions of pressure. The examined pressure and temperature dependence of the density describes this anomaly. If the fluid depth is insignificant, the pressure dependence of the density can be ignored. However, at large depths, in particular, in deep-water lakes (the maximum depth of Lake

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Baikal is 1637 m, and the average depth is 730 m), pressure gradients can have a significant effect on the density distribution, and hence, and on convective processes (see [4]).

Stability is studied as follows. The equations of motion in dimensionless variables are first derived. Then, the basic state (mechanical equilibrium in this case) is determined. Next, small perturbations are added to the functions describing the velocity, pressure, and temperatures distributions in the state of rest, and the problem is then written in terms of perturbations and is linearized. The linear problem for perturbations admits solutions of special form, in particular, in the case of exponential time dependences of perturbations (normal perturbations) [5]. This allows one to separate variables and obtain a spectral problem, which is studied numerically.

One of the best-studied convection models is the Oberbeck–Boussinesq approximation [5]. This model includes the Navier–Stokes equations and the heat-transfer equation with the density considered a liner function of temperature. In the derivation of the equations of this approximation, it is assumed that the density deviations from a certain average value due to fluid thermal expansion are so small that they can be ignored in all equations, except in the momentum equation, where these deviations are significant only in the terms involving the buoyancy force, which is responsible for the occurrence of convective motion. In the problem considered, this assumption is also adopted.

For the Oberbeck–Boussinesq approximation, the perturbation monotonicity principle (the principle of monotonic variation in stability) was proved [5]. This principle is as follows. The solutions of the linear problem for perturbations that arises in the stability analysis are sought in the form of the so-called normal perturbations $(\boldsymbol{V}', P', T')(x, y, z, t) \sim (\boldsymbol{V}, P, T)(z)\,\mathrm{e}^{\sigma t}$, where σ is a decrement that defines perturbation propagation with time. The eigenvalues σ of the corresponding spectral problem are generally complex) (such perturbations oscillate at a frequency determined by the imaginary part of the decrement). The damping or growth of these oscillating perturbations depends on the sign of the real part of the decrement. If $\mathrm{Re}\,\sigma < 0$, such perturbations damp with time and the initial state of the fluid is stable. The presence of perturbations with a positive real part of the decrement implies instability of the initial state with respect to these perturbations. The eigenvalues that lie on the imaginary axis are called critical (threshold) eigenvalues. In the parameter space, they separate the stability region from the instability region. In the Oberbeck–Boussinesq approximation, all eigenvalues remain real (in this case, perturbations vary monotonically with time) and the critical Rayleigh numbers are determined from the condition $\sigma = 0$. It is assumed that for the given model, too, the analysis can be confined to monotonic instability.

Investigation of penetrative convection in a horizontal fluid layer with density inversion was first performed in [6], together with an analysis of experimental studies [7] of convection in a horizontal water layer heated from above, whose lower boundary was maintained at a temperature equal to 0°C. The following facts were established in the experiments: convection was observed not only in the instability region but also in the zone of stable water stratification; over the entire convection region, except near the upper and lower boundaries, the water temperature was about 4°C; in the case where the temperature of the upper boundary was in the range of 12–20°C, additional convective cells on the vertical arose.

In [6], a layer with free isothermal boundaries was studied. In the vicinity of the temperature of maximum density, the equation of state for water was given by the relation $\rho = \rho_0(1 - \gamma \Delta T^2)$, where ΔT is the temperature deviation from the inversion temperature, ρ_0 is the maximum density, and $\gamma = 7.68 \cdot 10^{-6}\,{}^\circ\mathrm{C}^{-2}$. For the spectral problem in the case of free boundaries, the perturbation monotonicity principle was formulated. Linear stability analysis showed that at a layer surface temperature $T = 4^\circ\mathrm{C}$ the critical Rayleigh number coincided with its value in the Rayleigh–Benard problem and decreased with a temperature rise. In [6], the following explanation of this interesting fact was proposed: the stable upper layer compensated for the boundary conditions, which allowed the perturbations to take the most convenient shape. The minimum of the critical Rayleigh number was reached at a surface temperature equal to 6.7°C. In addition, the author noted an analogy with the stability problem for Couette flow between oppositely rotating cylinders in the case where the boundaries of the layer are considered solid. This allowed the critical Rayleigh number to be calculated from known results for this case, too. Later, this analogy was investigated in other studies (see, for example, [8]). In [6], a penetrative convection model in the Oberbeck–Boussinesq approximation was used.

The applicability of the Oberbeck–Boussinesq approximation in some problems related to the occurrence of convection in fluid layers is considered in [9, 10] taking into account additional factors characterizing thermal expansion.

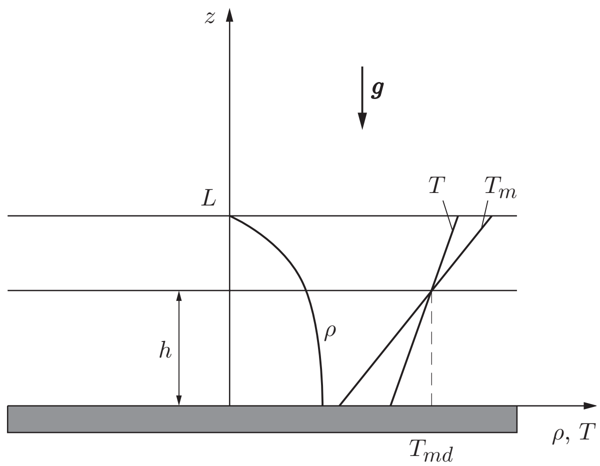

Fig. 1. Density and temperature distributions in mechanical equilibrium.

A review of the numerical methods used to solve hydrodynamic stability problems is given in [11]. In numerical studies of linear eigenvalue boundary-value problems with boundary conditions imposed at both ends of the integration interval, the solution of the original problem can be reduced to successive solutions of a number of derivative problems with boundary conditions specified only at one end of the interval. Each of these problems is derived from the original problem, and the solution of the original problem is sought as a linear combination of the solutions of these derivative problems. The coefficients of the linear combination are determined from the boundary conditions of the original problem, and the eigenvalues are zeroes of a certain test function. More details of this approach and some of its modifications and special features of use are described in [12–14].

Data of full-scale observations at Lake Baikal used in numerical calculations are given in [2, 15]. In particular, it is noted that the mesothermic maximum density point (the point at which the temperature coincides with the inversion temperature) in the lake is located at a depth of 200–300 m. Data on changes in the temperature regime are also given in [2, 15].

1. Formulation of the Problem. We consider an infinite horizontal layer of an initially quiescent fluid of thickness L with planar boundaries at a constant temperature. The lower boundary of the layer is considered solid, and the upper boundary is a free nondeformable surface. The temperatures on the lower and upper boundaries are equal to T_1 and T_2, respectively. The fluid (water) density varies as [16]

$$\rho(T,p) = \rho_m(p)[1 - \gamma(p)(T - T_m(p))^2], \tag{1}$$

where

$$\rho_m(p) = \rho_0(1 + e_\rho p) = 999.972 + 4.916021 \cdot 10^{-2} p,$$

$$\gamma(p) = \gamma_0(1 - e_\gamma p) = 8.572628 \cdot 10^{-6} - 7.061491 \cdot 10^{-9} p,$$

$$T_m(p) = T_0(1 - e_T p) = 3.985694 - 0.020617p$$

(the pressure in bars and the temperature in Celsius). In the temperature range from 0 to 10°C, the inaccuracy of this formula does not exceed 0.006%.

The mesothermic density maximum is reached in the layer at the inversion temperature T_{md}. For definiteness, we set $T_1 < T_{md} < T_2$. The coordinate system is introduced so that the x and y axes are in the plane of the lower boundary of the layer and the basis vector of the z axis is directed from the lower to the upper boundary. The coordinate system and the density and the temperature distributions in the layer in equilibrium are shown in Fig. 1.

In the limiting case where the functions $\rho_m(p)$, $\gamma(p)$, $T_m(p)$ are constants, the problem of penetrative convection in a similar formulation was studied in [8].

2. Equations of Motion in Dimensionless Variables. In the derivation of the convection model, the basic equations are the Navier–Stokes equations and the heat-transfer equation. As in the Oberbeck–Boussinesq approximation, the temperature and pressure dependence of the density is manifested only in the terms involving the buoyancy force. The temperature is reckoned from the lower-boundary temperature.

The following dimensionless quantities are introduced:

$$\rho = R\rho', \qquad T = \theta T', \qquad \boldsymbol{x} = h\boldsymbol{x}', \qquad \boldsymbol{V} = \vartheta \boldsymbol{V}', \qquad p = Pp', \qquad t = \tau t'.$$

As the characteristic scales, we choose the maximum density on the surface of the layer ρ_0, the width h of the part of the layer below the inversion point, the temperature difference $T_0 - T_1$, and the velocity of convective rise of a heated fluid particle $\vartheta = \sqrt{gh\gamma_0\theta^2}$. The pressure and time scales are expressed in terms of the chosen quantities as $P = \rho_0\vartheta^2$ and $\tau = h/\vartheta$ (see [17]). We note that instability can arise both in the region below the inversion point and in the upper part of the layer. The thickness of the lower part is always used as the characteristic length scale to combine these two cases. This choice is justified when the inversion point is located closer to the middle of the layer, i.e., both parts of the layer are comparable in thickness.

Under the above assumptions, the free-convection equations and the boundary conditions are written in dimensionless variables as follows (primes are omitted):

$$\begin{gathered}
\frac{d\boldsymbol{V}}{dt} = \nu\Delta\boldsymbol{V} - \nabla p - \frac{1}{\beta}(1+\varepsilon_\rho p)[1-\beta(1-\varepsilon_\gamma p)(T-1+\varepsilon_T p)^2]\boldsymbol{k},\\
\frac{dT}{dt} = \delta\Delta T, \qquad \nabla\cdot\boldsymbol{V} = 0,\\
z = 0: \qquad T = 0, \quad \boldsymbol{V} = 0,\\
z = \lambda: \qquad T = m\lambda, \quad v_z = \frac{\partial v_x}{\partial z} = \frac{\partial v_y}{\partial z} = 0.
\end{gathered} \tag{2}$$

Here $\boldsymbol{V} = (v_x, v_y, v_z)$ is the velocity, T is the temperature deviation from the lower-boundary temperature, p is the pressure, $\boldsymbol{k}$ is the basis vector of the z axis, and $d/dt = \partial/\partial t + (\boldsymbol{V}\cdot\nabla)$.

In the problem, the following dimensionless parameters appear: $\nu = \eta/(\rho_0\vartheta h)$ is the kinematic viscosity parameter, $\beta = \gamma_0\theta^2$, $\varepsilon_\rho = e_\rho P$, $\varepsilon_\gamma = e_\gamma P$, $\varepsilon_T = T_0 e_T P/\theta$, $\delta = \varkappa/(c_p\rho_0\vartheta h)$ is the Fourier number, $m = (T_{md} - T_1)/(T_0 - T_1)$, $\lambda = L/h$ is the inversion parameter that characterizes the position of the mesothermic maximum density point in the layer [by virtue of the linearity of the equilibrium temperature distribution, $\lambda = (T_2 - T_1)/(T_{md} - T_1)$]. Here η and $\varkappa$ are the dynamic viscosity and thermal conductivity, and c_p is the specific heat of the fluid.

Since we study instability only with respect to monotonic perturbations, we shall consider steady-state solutions of the problem that are periodic in x and y.

3. Basic State (Mechanical Equilibrium). The steady-state solution $\boldsymbol{V}_0$, p_0, T_0 of the boundary-value problem (2) that correspond to mechanical equilibrium is written as

$$\boldsymbol{V}_0 = 0, \qquad T_0 = mz.$$

The pressure is determined by numerical solution of the equation

$$\begin{aligned}
\frac{dp_0}{dz} = {}& -\varepsilon_\rho\varepsilon_\gamma\varepsilon_T^2 p_0^4 + [-2\varepsilon_\rho\varepsilon_\gamma\varepsilon_T(mz-1) + \varepsilon_T^2(\varepsilon_\rho - \varepsilon_\gamma)]p_0^3\\
& + [-\varepsilon_\rho\varepsilon_\gamma(mz-1)^2 + 2\varepsilon_T(\varepsilon_\rho - \varepsilon_\gamma)(mz-1) + \varepsilon_T^2]p_0^2\\
& + \Big[(\varepsilon_\rho - \varepsilon_\gamma)(mz-1)^2 + 2\varepsilon_T(mz-1) - \frac{\varepsilon_\rho}{\beta}\Big]p_0 + (mz-1)^2 - \frac{1}{\beta}
\end{aligned}$$

for $p_0 = 0 \quad (z = \lambda)$.

4. Linearized Equations for Perturbations. New steady-state periodic solutions of the problem (2) are sought in the form $\bar{\boldsymbol{V}} = \boldsymbol{V}_0 + \delta\boldsymbol{V}$, $\bar{p} = p_0 + \nu\delta p$, $\bar{T} = T_0 + T$, where $\boldsymbol{V}$, p, and T are unknown perturbations. Taking into account the expressions for $\boldsymbol{V}_0$, p_0, and T_0 after linearization, we obtain the equations for velocity, pressure, and temperature perturbations

$$\Delta\boldsymbol{V} - \nabla p + (\zeta(z)\mathrm{R}T + \xi(z)p)\boldsymbol{k} = 0,$$

$$mv_z = \Delta T, \tag{3}$$

$$\nabla \cdot \boldsymbol{V} = 0$$

with the corresponding boundary conditions

$$\begin{aligned} z = 0&: \qquad T = 0, \quad \boldsymbol{V} = 0, \\ z = \lambda&: \qquad T = 0, \quad v_z = \frac{\partial v_x}{\partial z} = \frac{\partial v_y}{\partial z} = 0. \end{aligned} \tag{4}$$

Here $\zeta(z) = (1 + \varepsilon_\rho p_0)(1 - \varepsilon_\gamma p_0)f$, $f = T_0 - 1 + \varepsilon_T p_0$, $\xi(z) = 2\varepsilon_T \zeta(z) + \varepsilon_\rho(1 - \varepsilon_\gamma p_0)f^2 - \varepsilon_\gamma(1 + \varepsilon_\rho p_0)f^2 - \varepsilon_\rho/\beta$, and $\mathrm{R} = 2/(\nu\delta)$ is the Rayleigh number.

5. Spectral Problem for Determining the Critical Rayleigh Number. We seek solutions of the linear problem (3), (4) in the form of normal perturbations:

$$(\boldsymbol{V}', p', T')(x, y, z, t) = (\boldsymbol{V}, p, T)(z) \exp{(\sigma t + i\alpha_x x + i\alpha_y y)}.$$

By virtue of the assumption of monotonic instability, we set $\sigma = 0$. Separating variables and eliminating pressure, we obtain the following eigenvalue boundary-value problem for an ordinary differential equation:

$$T^{(6)} - \xi(z)T^{(5)} - 3\alpha^2 T^{(4)} + 2\alpha^2\xi(z)T^{(3)} + 3\alpha^4 T'' - \alpha^4\xi(z)T' - (\alpha^6 + m\alpha^2\zeta(z)\mathrm{R})T = 0; \tag{5}$$

$$\begin{aligned} z = 0&: \qquad T = T'' = T^{(3)} - \alpha^2 T' = 0, \\ z = \lambda&: \qquad T = T'' = T^{(4)} = 0. \end{aligned} \tag{6}$$

Here $\alpha^2 = \alpha_x^2 + \alpha_y^2$ is the wavenumber.

6. Algorithm of Numerical Solution of the Spectral Problem. The eigenvalue problem (5), (6) is studied numerically as follows.

The temperature perturbation amplitude is written as the linear combination

$$T = \sum a_k w_k(z), \qquad k = \overline{1, 6}, \tag{7}$$

where each of the functions $w_k(z)$ is a solution of the Cauchy problem

$$\begin{gathered} Lw_k(z) = 0, \\ w_k(\lambda) = 0, \\ \dots\dots\dots\dots \\ w_k^{(k-1)}(\lambda) = 1, \\ \dots\dots\dots\dots \\ w_k^{(5)}(\lambda) = 0 \end{gathered} \tag{8}$$

[L is a differential operator that corresponds to Eq. (5)].

For $z = \lambda$, the boundary conditions imply that $a_1 = a_3 = a_5 = 0$. Using the boundary conditions for $z = 0$, we obtain a homogeneous system of linear algebraic equations for the other three coefficients of the linear combination (7). This system has a nontrivial solution if its determinant is equal to zero. For the specified value of α, the condition of vanishing of the determinant of this system determines the critical Rayleigh number. It should be noted that sometimes because of the complex dependence of the characteristic determinant on the parameters, it is difficult to find its zeros. Then, instead of seeking the zeros of the determinant, it is reasonable to require that one of conditions (6) be satisfied. In these cases, any of the nonzero physical quantities is normalized so that its value remains constant as the problem parameters are changed. Then, the coefficients of the series (7) can be determined from the normalization condition and the remaining boundary conditions [14]. The minimum value obtained is taken to be the true critical value of the Rayleigh number R_*. This determination is called the Rayleigh principle.

This sequence of operations, however, can lead to a loss of accuracy in numerical calculations. Indeed, the solution of the Cauchy problem (8) is equivalent to the solution of a system of linear ordinary differential equations for which the vector of unknown quantities includes the function $w_k(z)$ and the first five of its derivatives. During numerical solution, as the end of the integration interval is approached, the system of the vectors into which the solution of the problem (5), (6) is expanded can degenerate, resulting in an increase in the error in determining the coefficients of the linear combination (7) and the solution of the original problem at intermediate points. To avoid this, it is common to use the approach described, for example, in [12, 13]. The integration interval is broken up into a number of intervals of smaller length, and the system of basis vectors is orthogonalized at the stop points and is then used as the initial data for the integration in the next interval. The coefficients of the series (7) are determined for each of these intervals by recursive formulas. The orthogonalizations thus performed prevent flattening of the basis vectors, and the calculation accuracy increases.

7. Calculation of Critical Flows. Critical flows satisfy the steady-state equations (3) and (4). For their analysis, it is convenient to introduce the stream function for the velocity perturbations $\psi(x,z)$ [the problem is now solved on the plane (x,z)] using the formulas

$$v_x = \frac{\partial\psi(x,z)}{\partial z}, \qquad v_z = -\frac{\partial\psi(x,z)}{\partial x}.$$

Let us consider normal perturbations which are periodic in x:

$$\psi'(x,z) = \psi(z)\,\mathrm{e}^{i\alpha_x x}, \qquad p'(x,z) = p(z)\,\mathrm{e}^{i\alpha_x x}, \qquad T'(x,z) = T(z)\,\mathrm{e}^{i\alpha_x x}.$$

Introducing the stream function into (3), (4) and proceeding further as in the case described above, we obtain the boundary-value problem for the temperature perturbation, which has the same form as the problem (5), (6) by virtue of invariance under rotation of the horizontal plane. However, the wavenumber α is now given by the formula $\alpha = \alpha_x$. In this case, the stream-function perturbation amplitude $\psi(z)$ is related to the temperature perturbation amplitude $T(z)$ by the formula

$$\psi(z) = (\alpha^2 T(z) - T''(z))/(i\alpha m),$$

and the streamlines are the surface isolines

$$F(x,z) = \mathrm{Re}\,[\psi'(x,z)]. \tag{9}$$

In the numerical calculations, the values of the Rayleigh number and the wavenumber were set equal to the corresponding critical values.

8. Numerical Experiments. To calculate the dimensionless quantities in the spectral problem (5), (6), it is sufficient, for example, to specify the layer thickness L, the inversion parameter λ, and the lower-boundary temperature T_1 and to use the values of the physical constants from the equation of state (1). In determining the function describing the temperature distribution in rest, one should also find the value of the parameter $m = (T_{md} - T_1)/(T_0 - T_1)$, i.e., the point T_{md} of intersection of the temperature profile for the equilibrium state with the inversion-temperature curve. For this, one can use the known pressure dependence of the inversion temperature given by the formula $T_m(p) = T_0(1 - e_T p) = 3.985694 - 0.020617p$. Because the density deviations from the average value due to pressure and temperature variations are small, in this formula one can use hydrostatic pressure, i.e., the pressure corresponding to equilibrium at a constant characteristic density. We note that a linear dependence of the inversion temperature on depth adequately describes the behavior of the water inversion temperature. For example, in [2], the formula $T_m = 3.98 - 0.0021H$ is given, where H is the depth in meters.

In the calculation of the streamlines shown in Fig. 2, the temperature at the bottom of the layer and the position of the inversion point were chosen to be close to the data of full-scale observations at Lake Baikal [2, 15]. The temperature of the lower boundary was set equal to 3°C, the inversion point was at a depth of 300 m, and $L = 500$, 730, and 1000 m; the position of the inversion point corresponds to the value $z = 1$. The stream-function perturbation amplitude is determined after the solution of the homogeneous system of linear algebraic equations for the coefficients of the series (7) with accuracy up to an arbitrary factor. The isoline values in Figs. 2 and 3 are given for the case where the norm of the solution vector of this system is equal to 100.

The occurrence of instability in the lower part of the layer is shown in Fig. 3. This picture is less characteristic of Lake Baikal, but it is possible to imagine a situation where at the bottom of the lake there is a source that causes convective motion. In Fig. 3, the depth of the layer and the temperature at the bottom are the same and are 730 m and 1°C, respectively; the position of the inversion point changed. For intermediate values of the parameter λ, as in Fig. 2b, the occurrence of two convective cells is possible.

From Figs. 2 and 3, it follows that the position of the inversion point has a significant effect on the flow pattern in the layer. The fluid particles from the surface of the layer can penetrate to a depth of about hundreds

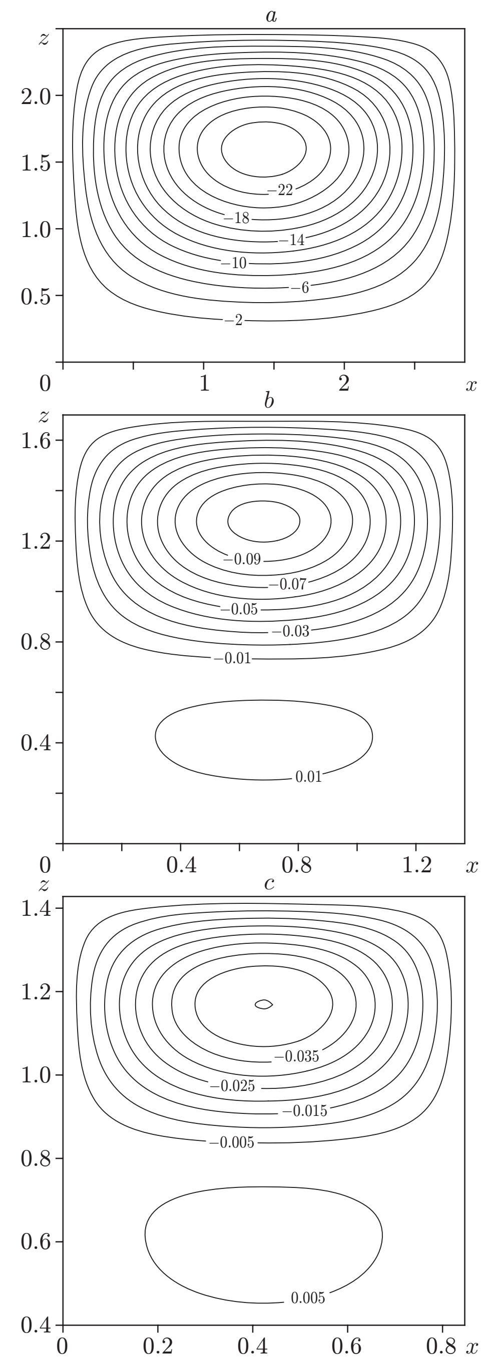

Fig. 2

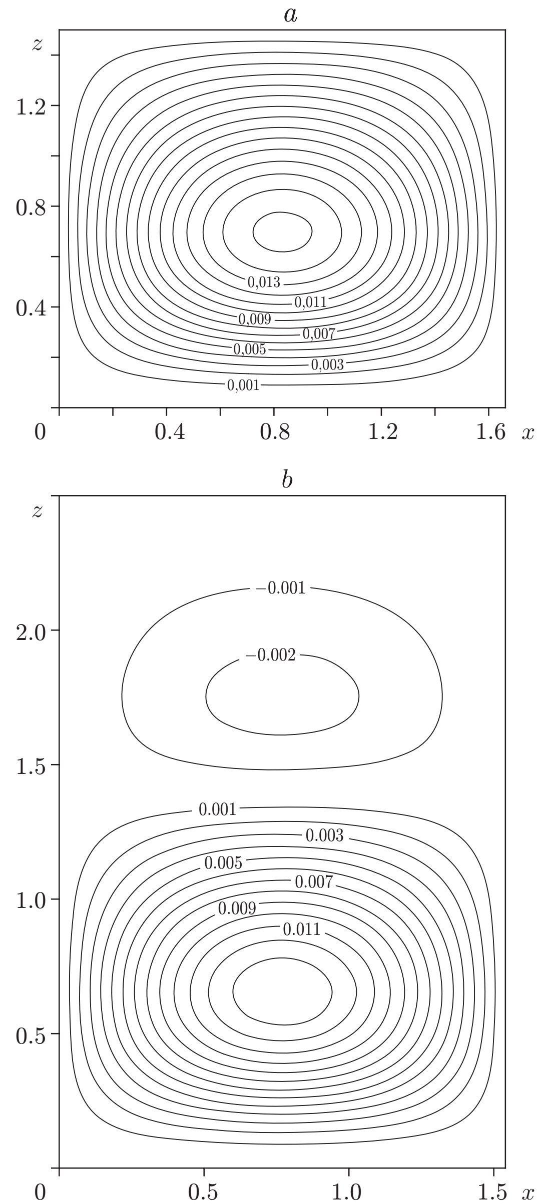

Fig. 3

Fig. 2. Occurrence of instability in the upper part of the layer: (a) $L = 500$ m, $T_1 = 3°$C, $T_{md} = 3.38°$C, and $T_2 = 3.95°$C; (b) $L = 730$ m, $T_1 = 3°$C, $T_{md} = 3.38°$C, and $T_2 = 3.64°$C; (c) $L = 1000$ m, $T_1 = 3°$C, $T_{md} = 3.38°$C, $T_2 = 3.54°$C; the numbers are values of the surface isolines (9).

Fig. 3. Occurrence of instability in the lower part of the layer: (a) $L = 730$ m, $T_1 = 1°$C, $T_{md} = 3.49°$C, $T_2 = 4.74°$C, and $\lambda = 1.5$; (b) $L = 730$ m, $T_1 = 1°$C, $T_{md} = 2.93°$C, $T_2 = 7.76°$C, and $\lambda = 3.5$; the numbers are values of the surface isolines (9).

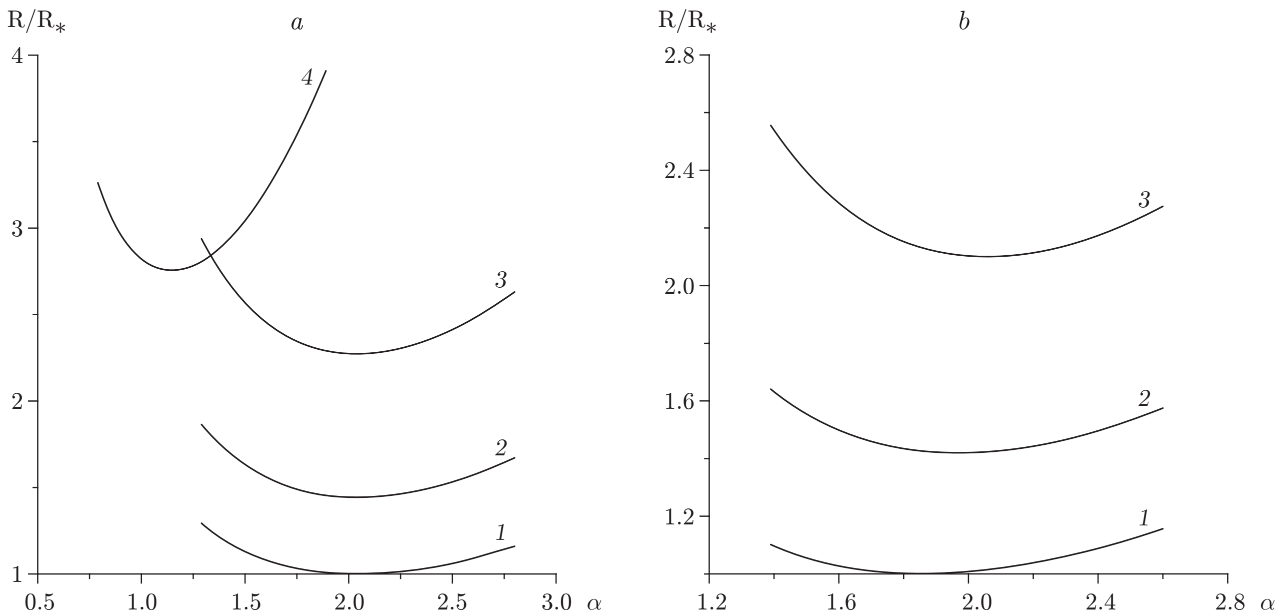

Fig. 4. Neutral curves: (a) $T_1 = 2°$C, $\lambda = 2.4$, and $L = 1$ (1), 200 (2), 400 (3), and 1070 m (4); (b) $T_1 = 1°$C, $T_2 = 6.08°$C, and $L = 1$ (1), 250 (2), and $L = 500$ m (3).

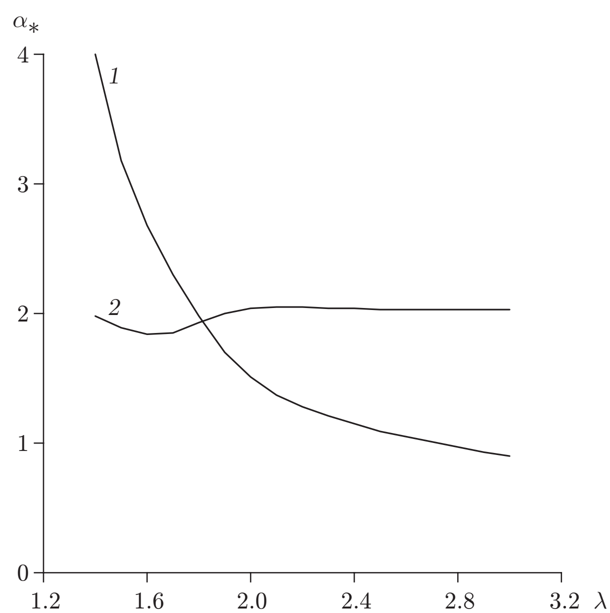

Fig. 5. Critical wavenumber versus the inversion parameter: 1) the temperature gradient on the boundaries of the layer is lower than the inversion temperature gradient; 2) the temperature gradient on the boundaries of the layer is higher than the inversion-temperature gradient.

of meters. In the case considered, this can be explained by the fact that the linear temperature profile is unstable above the inversion point (see Fig. 2) and below it (see Fig. 3) [18].

Curves of the quantities describing stability (critical Rayleigh number, critical wavenumber) versus problem parameters are given in Figs. 4 and 5.

Figure 4 shows neutral curves (curves of the Rayleigh number versus wavenumber); the values of the Rayleigh number are normalized by the corresponding limiting critical value R_* [8]. It can be noted that if the depth of the layer is insignificant (in such cases, pressure gradients have little effect and the functions ρ_m, γ, and T_m in Eq. (1) can be considered constant, the critical values of the Rayleigh number (which are chosen as minima on the corresponding neutral curves) agree well with the values for the limiting case. For large values of L, significant differences arise.

In Fig. 4a, curve 4 is of interest. The value of the critical wavenumber differs from the corresponding limiting value [8]. The problem can be considered for the two cases where the temperature gradient on the boundaries of the layer is higher or lower than the corresponding gradient of the inversion temperature. [A change in the depth of the layer results in a change in the inversion-temperature gradient on the layer boundaries (due to an increase in the pressure gradient) and a change in the difference $T_2 - T_1$ (due to the linearity of the temperature distribution in the state of mechanical equilibrium)]. The values of α_* for various values of the inversion parameters in these two cases are shown in Fig. 5. A change in the critical wavenumber α_* compared to the limiting value [8] is observed for fairly low temperature gradients. This case is shown by curve 1. In the other case, the critical value of the wavenumber coincides with its limiting value (curve 2).

Conclusions. A convection model taking into account the temperature and pressure dependence of the fluid density was considered. The numerical analysis of this model leads to the following conclusions.

The pressure dependence of the density has a significant effect on convection in deep fluid layers.

For a small depth of the layer and fairly small values of the parameters ε_ρ, ε_γ, and ε_T, which characterize the pressure dependence of the density, this dependence can be ignored. In such cases, the results are in good agreement with the results known for the case of penetrative convection.

The processes due to the anomaly of thermal expansion can be responsible for convection. The motion that arises in this case is capable of transferring particles from the surface to a depth of several hundred meters. However, it should be taken into account that the real temperature distribution is not linear, according to data of full-scale observations.

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